Theoretical Analysis of Coherent Optical FSK Systems with Limiter-Discriminator Detection

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Abstract—A detailed theoretical analysis, based upon Rice's theory of clicks, for evaluating the error performance of heterodyne optical FSK receivers with limiter-discriminator detection and pre- and integrate-and-dump (I&D) post-detection filtering is developed, accounting for laser phase noise, intersymbol interference (ISI) and intermediate-frequency (IF) filtered carrier envelope fluctuation effects.

From the analysis it turns out that the envelope fluctuations induced by phase noise may produce, for operating IF bandwidths, a much smoother transition to the bit error rate floor than expected when considering IF infinite bandwidth, thereby generating performance degradation right in the operating signal-to-noise ratio (SNR) region. The contribution to this degradation due to a change in the click statistics generated by the envelope fluctuations is found to be negligible. From the analysis, it is also found that the floor gets lower and approaches more rapidly as the modulation index and the bandwidth get larger, until the known floor for infinite IF bandwidth is reached.

Finally, general curves are presented for system performance relating bit error probability to optimum IF bandwidth, optimum modulation index, laser linewidth and receiver sensitivity. These curves represent the ultimate limit to performance of a heterodyne optical FSK receiver with limiter-discriminator detection and pre- and I&D post-detection filtering.

I. INTRODUCTION

The analytical evaluation of the performance of an FSK receiver with limiter-discriminator detection and pre- and integrate-and-dump (I&D) post-detection filter has been the subject of several studies. In the absence of carrier phase noise a complete solution to the problem has been given in [1] and [2], using Rice's theory of FM clicks [3].

In coherent optical FSK systems the presence of laser phase noise makes the analytical evaluation of the performance considerably more complicated than in the RF case. In [4] the behavior of an optical M-ary FSK system with standard matched filtering, which is optimum with ideal lasers (no phase noise), is analyzed. The approach taken there is quite simple but clever, and for the binary case it can be outlined as follows. At low signal-to-noise ratios the shot noise is the main source of errors, so the expression for bit error probability can be approximated by the classical result

$$P_b = \frac{1}{2} \exp \left( -N_{sea} / 2 \right),$$

where $N_{sea}$ is the effective number of detected photons/bit taking into account the phase noise signal suppression effect [4]. At high signal-to-noise ratios the phase noise is the dominant source of errors, and an error rate floor is generated bounded by

$$P_{f_{err}} < \frac{1}{2} \exp \left( -N_{sea} / 2 \right),$$

where $N_{sea} = \pi \hbar / 2 \Delta v T$ with $h$ being the modulation index and $\Delta v T$ the laser linewidth normalized to the bit rate $1/T$. Having the two expressions identical form, in [4] an estimate for BER valid for all signal-to-noise ratios is given utilizing

$$N_S = N_{sea} N_{sea}/(N_{sea} + N_{sea})$$

in place of $N_{sea}$ or $N_{sea}$ in the above expressions. The results obtained with this technique are then compared with those from Monte Carlo simulation, showing good agreement for small tone spacing.

Performance of a heterodyne optical MSK system with limiter-discriminator and I&D filter is also analyzed in [5]. The analysis presented there does make use of the clicks theory, but the effect of laser phase noise on the statistics of relevant system parameters such as clicks is neglected without justification, and neither the ISI introduced by the IF filter nor filtered signal envelope fluctuation effects are considered.

Based on the approach proposed by Foschini, Greenstein and Vannucci [6]–[8], an interesting bit error rate analysis for FSK coherent optical receivers with delay and multiply detection has recently appeared in [9]. In this paper the analysis is developed only for an ideal integrator IF filter, and clicks are not of concern as delay and multiply detection is intrinsically not affected by this phenomenon.

So, although results from simulation, estimates and limited analyses exist in the literature for optical FSK receivers, none accounts at the same time for phase noise, ISI and signal and phase noise FM-AM conversion when ideal limiter-discriminator and both pre- and post-discrimination filters are considered. In this paper we present a detailed analysis of this situation, based on Rice's theory of clicks and some new results on click statistics with phase noise obtained in [10], with the aim of identifying the effect of the individual sources of impairments, validating, if the case, previous more limited analyses, and relating in general bit error probability and optimal IF bandwidth to modulation index, laser linewidth and receiver sensitivity. The system impairments taken into account are: i) the shot noise produced by the photodetector which is modeled by a white Gaussian noise; ii) the laser phase noise modeled by the integral of a white Gaussian noise; iii) the phase and envelope distortion produced by the pre-detection filter; iv) the subsequent pattern-dependent intersymbol interference arising in the detected signal.

As the analysis is carried on through several steps and approximations, we find it useful to provide, already in the introduction and even before formally stating the problem, a list of the main
The local oscillator (LO) laser generates the optical waveform
\[ s_{LO}(t) = \sqrt{2P_{LO}} \cos [\omega_{LO}t + \phi_{LO}(t)] \]  
where \( P_{LO} \) is the LO optical power, \( \phi_{LO}(t) \) is the LO phase noise. The LO laser has a nominal frequency \( f_{LO} \) differing from the carrier frequency \( f_c \) by the intermediate frequency (IF) \( f_{IF} \):
\[ f_{IF} = |f_{LO} - f_c| \]  
Precise setting and stabilization of the laser frequencies is obtained at the transmitter and the receiver by properly setting the bias current and accurately stabilizing the temperature. Fine control over \( f_{IF} \) is achieved at the receiver by an automatic frequency control (AFC).

After combining signal and local oscillator fields, the resulting field is photodetected and amplified for subsequent processing. Photodetection is normally performed through a balanced optical front-end, consisting of two photodiodes with proper configuration. This scheme greatly reduces the effect of random intensity fluctuations of the LO laser (RIN - Relative Intensity Noise), which otherwise would reduce substantially the system performance. After photodetection, low-noise preamplification is required.

Assuming perfect equalization of the receive front-end impulse response and shot-noise-limited operation, the input voltage to the IF filter (IF strip) can be written as
\[ v(t) = V \cos[\omega_{IF}t + \phi_S(t) + \phi_N(t)] + n(t) \]  
where \( V = 2R \sqrt{P_S P_{LO}} \), with \( R \) being the photodetector's responsivity, \( \phi_S(t) = \phi_{NT}(t) - \phi_{LO}(t) \) is the overall phase noise of the transmitting and LO lasers; \( n(t) \) is the photodetection shot-noise which, under the strong LO power assumption, may be considered white Gaussian with one-sided spectral density \( N_0 = 2eR P_{LO} \), with \( e \) being the electron charge.

In this paper we assume that the signal is
\[ \phi_S(t) = 2\pi f_d \int b(\tau) d\tau, \]  
where \( f_d \) is the frequency deviation and \( b(t) \) is an NRZ (Non-Return-to-Zero) pulse train with statistically independent binary symbols. This corresponds to assuming that the laser frequency response to frequency modulation is uniform over the modulation bandwidth. As this may not be true in semiconductor lasers, where frequency modulation is achieved by modulating the drive current, some equalization of the laser FM characteristics is assumed.

The spectral density of laser frequency noise has a \( 1/f \) portion at low frequencies and a flat part extending up to a few GHz, then, after a resonance peak, it goes down. The \( 1/f \) portion can be tracked out with the AFC, so the frequency noise can be treated as white as long as the resonance peak falls away from the frequencies of interest. Indeed the accepted model for frequency noise is white and Gaussian with two sided spectral density \( \Delta \nu = \Delta \nu_t + \Delta \nu_r \), being the sum of transmitting and receiving lasers' linewidths.

The IF bandpass filter \( h_{IF}(t) \) is taken to be symmetric around \( f_{IF} \) with low-pass equivalent transfer function \( H(f) \). \( H(f) \) has
per bit, and \( x(t) \) is the envelope of the filtered signal before limiting

\[
x(t) = \int_{-\infty}^{\infty} h(\tau) e^{i \phi(t-\tau)} d\tau
\]

In this paper the limiter is followed by an ideal discriminator, which simply takes the derivative of the instantaneous phase in (6), and a I&D post-detection filter to generate a waveform suitable for sampling and thresholding. Signal expression at the output of the I&D filter will be given in Section V when evaluating output noise statistics.

### III. IF Filtered Carrier Phase and ISI

Due to the presence of phase noise, a straightforward approach to the statistical description of \( \alpha(t) \) in (7) and \( x(t) \) in (11) is a formidable task. In such expressions the noise and the signal are mixed in such a nonlinear way that can discourage any further investigations. In the following, we use some results from the work of Bedrosian and Rice [11], who studied the output phase process resulting from filtering a signal with random phase with a symmetric bandpass filter, extended in [10] to the case of constant phase signal plus phase noise.

Bedrosian and Rice showed that \( \alpha(t) \) in (7) can be written as the sum of a term that represents linear filtering of the input phase and other terms representing various orders of distortion introduced by the filter

\[
\alpha(t) = \int_{-\infty}^{\infty} h(\tau) \phi(t-\tau) d\tau + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)!} f_{2n+1}(t)
\]

The explicit expressions for \( f_k(t) \) can be found in [11].

In Bedrosian and Rice's work a Gaussian input frequency process \( \phi(t) / 2\pi, \) white over a bandwidth smaller than the filter bandwidth, is considered. In this respect, when only phase noise is present in \( \alpha(t) \), our case is at the opposite extreme, being the frequency noise bandwidth much greater than the filter bandwidth, but analysis applies as well. Numerical results obtained as in [11] for the present case show that for small linewidths \( (\Delta \nu / B \ll 1) \) the term in (12) corresponding to linear filtering largely dominates also in our case. In Fig. 3 the mean square error (MSE) between the actual output frequency \( \alpha(t) / 2\pi \) and the linearly filtered one, represented by \( 1/2\pi \) times the time derivative of the first term in the right hand side of (12), is plotted as a function of \( \Delta \nu / B \). The MSE has been obtained by integration of the power spectral density of the dominant distortion term in expansion (12), whose expression is reported in [11] and also in (A.12).

The validity of the linear filtering approximation for the laser phase noise is also corroborated by the analysis in [13], where it is assumed to hold, on partly intuitive grounds, for filter bandwidths much greater than \( \Delta \nu \), and by the experiment in [14], which showed good agreement with the predictions of [13]. In [15] the results obtained in [13] and [14] are also related to the analysis in [11].

When a constant frequency signal is present in addition to phase noise, the authors have found in [10] that the linear filtering approximation of signal and phase noise is also acceptable for the same magnitude of normalized linewidths. Observe that for
this special case of constant frequency shift, the filter $h(t)$ in (12) can also be replaced by $h(t) = \cos 2\pi f_d t$ where $f_d$ becomes the frequency displacement of $H(f)$ [10].

As regard of signal portion in $\alpha(t)$ in the general case, it turns out that only two particular forms of $\phi_S(t)$ need actually be considered for performance evaluation, corresponding to the “all-one” and to the “alternating one-zero” bit patterns. Indeed, following Cartier [16] and Pawula [2], we observe that for time-bandwidth products $BT \geq 1$, with $B$ being the IF filter bandwidth, only the bits adjacent to that being detected need be taken into account for ISI evaluation. In fact, the results in [2] are in agreement with those in [1] where a 30-bit pattern is used. We therefore consider only the 111, 010, 011 and 110 cases, the other bits in the stream having no relevance. We can think of the 111 and 010 patterns as stemming from the “all-one” and “alternating one-zero” patterns, whereas the other two cases are viewed as resulting from one of the above patterns changing to the other in correspondence of the middle bit.

A. “All-One” Bit Pattern

In this case the phase $\phi_S(t)$ is a straight line and the output phase after the IF filter, except for a constant phase delay, is

$$\alpha_S(t) = 2\pi f_d t$$

and this is of course the same result we would obtain by linear filtering of the phase. As mentioned above, the linear approximation for this case of constant frequency signal is acceptable also in the presence of phase noise [10].

B. “Alternating One-Zero” Bit Pattern

In this case the phase $\phi_S(t)$ after FM modulation is a triangular pulse train, assumed with no loss of generality to pass through zero with positive slope for $t = 0$. Using the Fourier series expansion of $\phi_S(t)$

$$\phi_S(t) = \frac{4h}{\pi} \sum_{q=0}^{\infty} (-1)^q \sin 2\pi (2g+1) f_d t \frac{1}{(2g+1)^2}$$

where $h = 2f_d T$ and $f_i = 1/2T$, and truncating to the first two nonzero harmonics, under the condition $BT \leq 5$ to have the fifth harmonic suppressed by at least 3 dB, yields for the phase corresponding to linear filtering

$$\alpha_S(t) = \frac{4h}{\pi} \left[ |H(f_1)| \sin (2\pi f_i t + \angle H(f_1)) \right.$$  

$$\left. - \frac{|H(3f_1)|}{9} \sin (6\pi f_i t + \angle H(3f_1)) \right]$$

Expression (15) for $\alpha_S(t)$ is an approximation to that of (7). For wide bandwidth or low modulation index, however, the mean square error between (15) and (7), when only signal is present, is sufficiently small to let us adopt (15) instead of (7). For example, for a second order Butterworth filter we have found that the phase MSE is less than -20 dB for $h < 1.5$ when $BT = 1$ and even smaller for larger bandwidths. The corresponding frequency MSE also turns out to be less than -20 dB for all cases of practical interest.

When both altering one-zero bit pattern and phase noise are present, the evaluation of the MSE between the actual and the linearly filtered frequency is more involved. Following an approach similar to that in [10] and [11] with the modifications pertinent to the present case, we have derived formulas also for this signal pattern. As both derivation and formulas turn out to be very complicated, for simplicity the latter only are reported in Appendix A without derivation. The resulting frequency MSE error is shown in Fig. 4 vs. the modulation index $h$ for several values of $BT$ and $\Delta \nu / B$. Fig. 4 confirms the validity
of the linear approximation also in the present case, as for all combinations of \( h \) and \( BT \) of practical interest the MSE is below -20 dB.

For all the reasons discussed above we assume in our performance analysis that linear filtering approximation is valid in the presence of both signal and phase noise, so that the output phase from the IF filter will be from now on approximated by

\[ \alpha(t) = \int_{-\infty}^{\infty} h(\tau) \phi(t - \tau) d\tau \]  

(16)

IV. EFFECT OF IF FILTERING ON CARRIER ENVELOPE

Among other things, we are at last interested in finding the optimum IF bandwidth. Any finite IF bandwidth generates a conversion of both signal and noise phase fluctuations to envelope fluctuations that affect both the statistics of the clicks and the pdf of \( \theta(t) \) in (9). Even in the presence of a subsequent limiter this conversion should therefore be taken into account when evaluating performance.

Obtaining the statistics of the filtered envelope \( x(t) \) in (11) is not simple. An accurate approximation has been given by Foschini, Vannucci and Greenstein [6]-[8], but their results hold only in the absence of a phase signal, and it isn’t a straightforward matter extending them to the present case.

In this paper we approach the problem from a different point of view, by considering only two bit patterns, as described in Section III, and determining approximate expressions for \( x(t) \) in such two cases. As shown in [10, Appendix B] for the all-one bit pattern, and easily applicable to the alternating one-zero bit pattern, under some acceptable simplificative hypotheses, for \( \Delta \nu T \ll 1 \), using a Karhunen–Loève expansion for the phase noise yields the following results:

A. “All-One” Bit Pattern

\[ x(t) \simeq \sum_{n=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} H[f_d + (n + 3i)B/8] \\
\quad \cdot J_i(\beta_1)J_i(\beta_2)e^{i\pi(n+3i)^22Bt} \]

(17)

where \( J_k(a) \) are Bessel functions of first kind, of order \( k \) and argument \( a \), and \( \beta_k \) are independent Gaussian r.v.'s with zero mean and variance

\[ \sigma_{\beta_k}^2 = \frac{32\Delta \nu}{\pi B k^2} \quad k = 1, 3. \]  

(18)

B. “Alternating One-Zero” Bit Pattern

\[ x(t) \simeq \sum_{p,q,n,i=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} H[(p + 3q)f_1 + (n + 3i)B/8] \\
\quad \cdot J_p(\gamma_1)J_q(\gamma_3)J_n(\beta_1)J_i(\beta_3) \\
\quad \cdot e^{i2\pi[(p+3q)f_1+(n+3i)B/8]t} \]

(19)

where

\[ \gamma_1 = \frac{4h}{\pi} \]  

(20)

\[ \gamma_3 = \frac{4h}{9\pi} \]  

(21)

From (17) and (19) we conclude that, instead of \( p_X(x) \), the pdf's \( p_{\beta_k}(\beta) \), \( k = 1, 3 \), can be conveniently used for averaging out the r.v. \( x(t) \), whenever necessary. In [10] it has been verified that results generated by this technique are in excellent agreement with results obtained by averaging through accurate expressions of \( p_X(x) \), given in [6]-[8] and [17] for the short-term integrator with phase noise only.

V. NOISE STATISTICS

In this section we aim at determining the statistics of noise at the I&D filter output before thresholding. Being the discriminator output the derivative of the phase of (6), the subsequent I&D filter produces a phase difference \( \Delta \Phi(t) \) as follows

\[ \Delta \Phi(t) = \left[ \alpha_S(t) - \alpha_S(t - T) \right] \\
+ \left[ \alpha_N(t) - \alpha_N(t - T) \right] \\
+ \left[ \theta(t) - \theta(t - T) \right] \mod 2\pi \\
+ 2\pi N(t - T, t) \\
= \Delta \alpha_S(t) + \Delta \alpha_N(t) \\
+ \Psi(t) + 2\pi N(t - T, t) \]

(22)

where \( \Delta \alpha_S(t) \), \( \Delta \alpha_N(t) \) have evident meaning, \( \Psi(t) \) is a component of the output noise due to conversion of shot noise to phase noise

\[ \Psi(t) \triangleq \left[ \theta(t) - \theta(t - T) \right] \mod 2\pi \]

(23)

and \( N(t - T, t) = N(t) - N(t - T) \) is a discrete r.v. representing the net balance of positive and negative clicks [3], [12] in the time interval \( (t - T, t) \). The receiver decides a “1” was sent if, at the sampling time, \( \Delta \Phi > 0 \) and a “0” otherwise. The noise terms in (22) are \( \Delta \alpha_N(t) \), \( \Psi(t) \) and \( 2\pi N(t - T, t) \).

\( \Delta \alpha_N(t) \) is a stationary Gaussian process with zero mean and variance

\[ \sigma_N^2 = 2\pi \Delta \nu \int_{-\infty}^{\infty} H(f)^2 \left| \frac{\sin \pi f T}{\pi f} \right|^2 df \]

(24)

independent of the signal-to-noise ratio, and it is the cause of the error rate floor. As an example for a second order Butterworth filter it is \( \sigma_N^2 = 2\pi \Delta \nu T(1 - 1/(\pi \sqrt{2B}T)) \). For a \( T \)-sec short-term integrator, (24) gives \( \sigma_N^2 = 2\pi \Delta \nu T(1 - \gamma/(3T)) \). Interestingly, this latter expression provides for \( \sigma_N^2 \) values nearly equal to those obtained in [9] for a \( T \)-sec short-term integrator with \( T \)-sec delay demodulator, where it is found

\[ \sigma_N^2 = \pi \Delta \nu T(1 - (\gamma - \sin \gamma)/[\gamma(1 - \cos \gamma)]) + 2\pi \Delta \nu T(T - \tau) \]

with \( \gamma = \pi \tau/2T \).
The probability density function (pdf) of the r.v. \( \theta(t) \), conditioned on \( x(t) \), is [18]
\[
p_{\theta}(\theta|x) = \int_0^\infty \frac{y}{\pi} e^{-\left(y^2 + \rho x^2 - 2y\sqrt{\rho} \sqrt{\cos \theta} \right)} dy
\]
\[
= \frac{e^{-\rho x^2}}{2\pi} + \sqrt{\frac{\rho}{\pi}} x \cos \theta e^{-\rho x^2 \sin^2 \theta} Q(-\sqrt{2\rho} x \cos \theta)
\]
where \( Q(\alpha) \triangleq \int_{\alpha}^\infty \frac{1}{\sqrt{\pi}} e^{-t^2/2} dt \). The unconditional pdf of \( \theta(t) \) is
\[
p_{\theta}(\theta) = \int_0^\infty p_{\theta}(\theta|x)p_{X}(x) dx
\]  
(26)

\( p_{X}(x) \) being the pdf of \( x(t) \).

As regards of \( \Psi(t) \), the two r.v.'s \( \theta(t) \) and \( \theta(t - T) \) can be considered to be statistically independent [12], as they derive from the phase and shot noise and are spaced by a bit time. Then \( p_{\Psi}(\psi) \) is the convolution between two pdf's of the form (25) and can be written, generalizing the result given in [2] and [19] to allow for random envelope, as
\[
p_{\Psi}(\psi) = E_\beta \left\{ \frac{e^{-U}}{4\pi} \int_0^\pi dz \sin z 
\right.
\[
\left. \cdot \left[ 1 + U + V \cos z + W \sin z \cos \psi \right] \right. 
\]
\[
\left. \cdot \exp(V \cos z + W \sin z \cos \psi) \right\} |\psi| \leq \pi
\]  
(27)
in which
\[
U = \frac{1}{2} \rho \left[a_1^2 + a_2^2\right]
\]
\[
V = \frac{1}{2} \rho \left[a_1^2 - a_2^2\right]
\]
\[
W = \sqrt{U^2 - V^2} = \rho x_1 x_2
\]
and \( E_\beta \) means averaging over \( \beta_k \)'s. Notice that this pdf is pattern dependent through \( x_1 \) and \( x_2 \).

The last noise term in (22) is \( 2\pi N(t - T, t) \). In the absence of phase noise, \( N(t - T, t) \) would be Poisson with parameter [2]
\[
\Lambda = \int_{t-T}^t \frac{\hat{\alpha}(\tau)}{2\pi} e^{-\rho x^2(\tau)} d\tau
\]  
(28)

In the presence of phase noise, both \( \hat{\alpha}(t) \) and \( x(t) \) are random processes, and \( \Lambda \) in (28) becomes a r.v.. The process \( N(t - T, t) \) is therefore conditionally Poisson and the pdf of \( \Lambda \) is needed to evaluate its statistics. This pdf has been evaluated in [10] and the statistics of \( N(t - T, t) \) found. From the analysis in [10] it follows that for small phase noise \((\Delta \nu T \ll 1)\) we can neglect the phase noise term in \( \hat{\alpha}(t) \) in (28) with respect to the signal term \( \alpha_0(t) \), retaining only its effects through \( x(t) \). For a bit "1" being detected and \( \Delta \nu T \ll 1 \), \( N(t - T, t) \) takes only negative values, the probability of positive values being negligible and its distribution function, which depends on the bit pattern through \( x(t) \) and \( \alpha_0(t) \), turns out to be [10]
\[
P_N(k) = P\{N = -k\}
\]
\[
= E_\beta \left\{ \frac{e^{-\Lambda k}}{k!} \right\} k = 0, \ldots
\]  
(29)

where
\[
\Lambda = \int_{t-T}^t \frac{\alpha_0(\tau)}{2\pi} e^{-\rho x^2(\tau)} d\tau
\]  
(30)

We point out that in [10] (29) and (30) have been derived for a constant modulating signal such as the all-one pattern, but they are immediately extended to all kind of signals expandable in Fourier series, by considering that for these signals it is always possible to find an expression for the carrier envelope similar to (17) or (19) depending on any case on the two r.v.'s \( \beta_1 \) and \( \beta_2 \).

The three r.v.'s \( \Delta \alpha_N, \Psi \) and \( N \) can be considered to be statistically independent (see [12, p. 1518]). For bit "1" being detected, the pdf of the partial sum \( \mu \triangleq \Psi + 2\pi N \) in (22) can be written as
\[
p_{\mu}(\mu) = \sum_{k=0}^{\infty} p_{\Psi}(\mu + 2\pi k)P_N(k)
\]  
(31)

Having determined the statistics of all noise terms in (22), we evaluate in the next section the corresponding bit error probability.

**VI. BIT ERROR PROBABILITY**

The statistics of \( \mu \) depend on the bit pattern through \( x(t) \), \( \alpha_S(t) \) in \( P_N(k) \) and \( x(t - T) \) in \( p_{\Psi}(\mu) \), as said above. Given a certain bit pattern, the probability of a bit error can be evaluated as
\[
P_k(\cdots) = P\{\Delta \Phi < 0 | \cdots\}
\]
\[
= P\{\Delta \alpha_N + \mu < -\Delta \alpha_S | \cdots\}
\]
\[
= \int_{-\infty}^{\pi} p_{\mu}(\mu)Q\left(\frac{\mu + \Delta \alpha_S}{\sigma_N}\right) d\mu
\]  
(32)

where the dots "\( \cdots \)" stand for the pattern. Pattern dependence in (32) is introduced through the waveforms of \( x(t) \) and \( \alpha_S(t) \), which affect the statistics of \( \mu \) and the value of \( \Delta \alpha_S \). Letting
\[
\overline{\Lambda} \triangleq E_\beta \{\Lambda\}
\]  
(33)
\[
\overline{\Lambda^2} \triangleq E_\beta \{\Lambda^2\}
\]  
(34)

for small \( \overline{\Lambda} \) the above expression (32), noting that \( P_N(0) \approx 1 \), \( P_N(1) \approx 1 - P_N(0) \approx \overline{\Lambda} \) and \( 1 - P_N(0) - P_N(1) \approx \overline{\Lambda^2}/2 \), after some simple manipulations, can be put in a more significant form as follows.
\[ Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \]

\[ P_\mu(\mu) \]

\[ -5\pi < \mu < -\pi \]

\[ \Delta \alpha_S \]

\[ \mu, \Delta \alpha_S, \sigma_N \]

Fig. 5. Plot of integrand factors in (35) and (36).

For modulation indexes \( 0.5 \leq h \leq 1 \) it is \( |\Delta \alpha_S| < \pi \) and therefore, replacing \( Q((\mu + \Delta \alpha_S)/\sigma_N) \) by 1 for \( \mu < -\pi \) (see Fig. 5) yields the following upper bound

\[
P \{ \Delta \Phi < 0 \} \leq \int_{-\infty}^{-\pi} p_\mu(\mu) \, d\mu + \int_{-\pi}^{\pi} p_\mu(\mu) Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

\[
= 1 - P_N(0) + P_N(1) \int_{-\pi}^{\pi} p_\psi(\mu) Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

\[
< \overline{\lambda} + \int_{-\pi}^{\pi} p_\psi(\mu) Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

(35)

For \( 1 < h < 3 \), we have \( |\Delta \alpha_S| < 3\pi \) and replacing \( Q((\mu + \Delta \alpha_S)/\sigma_N) \) by 1 for \( \mu < -3\pi \) (see Fig. 5) yields now

\[
P \{ \Delta \Phi < 0 \} \leq \int_{-\infty}^{-3\pi} p_\mu(\mu) \, d\mu + \int_{-\pi}^{\pi} p_\mu(\mu) Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

\[
= 1 - P_N(0) - P_N(1)
\]

\[+ \int_{-\pi}^{\pi} [P_N(0)p_\psi(\mu) + P_N(1)p_\psi(\mu + 2\pi)] Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

\[
< \overline{\lambda}^2 + \int_{-3\pi}^{\pi} [p_\psi(\mu) + \overline{\lambda} \cdot p_\psi(\mu + 2\pi)] Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

(36)

The given pattern determines \( \Delta \alpha_S \) and the statistics of \( \mu \) from (31) through (27) and (29). The 011 and 110 bit patterns give for symmetry the same result. Thus, assuming the 011 pattern, each of the two patterns 111 and 010 can be considered to generate one half of the bit phase variation \( \Delta \alpha_S \) and of average number of clicks \( \lambda \) in (30) [2]

\[
\Delta \alpha_S(011) = \frac{1}{2} [\Delta \alpha_S(010) + \Delta \alpha_S(111)]
\]

(37)

\[
\overline{\lambda}(011) = \frac{1}{2} [\overline{\lambda}(111) + \overline{\lambda}(010)].
\]

(38)

Then the average bit error probability can be evaluated from (32) as

\[
P_b = \frac{1}{4} [P_b(111) + P_b(010) + 2P_b(011)].
\]

(39)

This is the final formula to be used for performance evaluation of the heterodyne optical FSK receiver with limiter-discriminator detection and pre- and I&D post-detection filtering.

As often done [2], we can identify a "click" contribution to the error probability \( P_b \) as distinguished from a "continuous" contribution

\[
P_b = P_{\text{click}} + P_{\text{cont}}
\]

(40)

For \( 0.5 \leq h \leq 1 \) the click part turns out to be

\[
P_{\text{click}} = \frac{1}{2} [\overline{\lambda}(111) + \overline{\lambda}(010)]
\]

(41)

while the continuous part is

\[
P_{\text{cont}} = \frac{1}{4} [P_{\text{cont}}(111) + P_{\text{cont}}(010) + 2P_{\text{cont}}(011)].
\]

(42)

with

\[
P_{\text{cont}}(\cdots) = \int_{-\pi}^{\pi} p_\psi(\mu) Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu
\]

(43)

Such a distinction can also be made when \( h > 1 \), if we let

\[
P_{\text{click}} = \frac{1}{2} [\overline{\lambda}^2(111) + \overline{\lambda}^2(010)]
\]

(44)

and \( P_{\text{cont}} \) again given by (42) with

\[
P_{\text{cont}}(\cdots) = \int_{-3\pi}^{\pi} [p_\psi(\mu) + \overline{\lambda} \cdot p_\psi(\mu + 2\pi)] Q \left( \frac{\mu + \Delta \alpha_S}{\sigma_N} \right) \, d\mu.
\]

(45)

But we point out that in this case the distinction is arbitrary as it is not possible to separate clearly the two contributions.

VII. NUMERICAL RESULTS AND COMMENTS

As the expressions of the terms in (39) are very complicated, calculation of the bit error probability \( P_b \) in (39) must be done by numerical methods. By using (13) and (17) for the "all-one" bit pattern and (15) and (19) for the "alternating one-zero" pattern in (30), the values of \( \overline{\lambda} \) in (33) and \( \overline{\lambda}^2 \) in (34) are readily found by numerical integration for the patterns 111 and 010; the values for the pattern 011 are the corresponding mean, as previously said. In (30) and in all other equations where an integral over a bit time is to be performed, in the case of the 010 pattern, integration is to be done between time instants corresponding to a minimum and a maximum of \( \alpha_S(t) \). By using (27), (43) and (45) are to be evaluated, again by numerical integration, for all three patterns of interest. The needed values of \( \Delta \alpha_S \) are easily obtained from (13), (15) as

\[
\Delta \alpha_S(111) = h \pi
\]

(46)

\[
\Delta \alpha_S(010) \simeq \frac{8h}{\pi} \left( |H(f_1)| + \frac{|H(3f_1)|}{9} \cos \delta \right)
\]

(47)

where

\[
\delta = \angle H(3f_1) - 3 \angle H(f_1).
\]

(48)

The above analysis has been applied to various kinds of IF filters, generating similar results and conclusions. For this reason we present here only the curves obtained for a second order Butterworth filter, shown in Fig. 6–13.
Before commenting on the results in detail, we would like to recall that in the absence of phase noise the limiter-discriminator-integrator detector is known to attain best performance when $h = 0.7$ and $BT = 1$ [1], [2]. Although possibly different from these latter values, it can be anticipated that optimum values of the IF bandwidth and modulation index do exist also in the presence of phase noise. In fact, the phase noise widens the signal spectrum forcing use of a larger bandwidth. We are thus allowed to increase the modulation index to combat the phase noise fluctuations. Since we cannot increase indefinitely the bandwidth, because we would get too much shot noise, or the modulation index, because the signal would be too much distorted, increasing the ISI, optimum IF bandwidth and modulation index should indeed exist.

The effects of IF bandwidth and modulation index on system performance can be observed from Fig. 6, 7 and 8. Fig. 6 refers to the case $h = 0.7$ and $BT = 1$ and shows the probability of bit error $P_b$ in (39) versus $E_b/N_0$ for various linewidths. For example, consider the curve labeled with $\Delta \nu T = 0.01$, which exhibits a bit error rate floor above $10^{-9}$. In Fig. 7, which is the same as Fig. 6 but with $BT = 2$, increasing the IF bandwidth has lowered the floor of the curve with $\Delta \nu T = 0.01$ below $10^{-14}$. Lastly, consider Fig. 8, which differs from Fig. 7 for a larger modulation index, i.e. $h = 1$. We can see that the floor lowers again and the curves tend to become steeper. We observed as a general trend that by increasing the bandwidth and the modulation index, the floor gets lower and is approached with a more pronounced knee in the curves.

When the bandwidth is sufficiently large and $\rho \to \infty$, an asymptotic expression for the floor can be derived from $P_{\text{cont}}$ in (43), which is the dominating term in (40). For $BT \gg 1$ and $\rho \to \infty$, in fact, $p_{\nu} (\mu)$ in (43) tends to a Dirac impulse, while

$$\Delta \alpha_s \to h \pi = 2\pi f_d T \quad \text{and} \quad \sigma_N^2 \to 2\pi \Delta \nu T,$$

so that

$$P_{b, \text{floor}} = P_{\text{cont}} = Q \left( \frac{\Delta \alpha_s}{\sigma_N} \right) = Q \left( f_d \sqrt{\frac{2\pi T}{\Delta \nu}} \right)$$

Equation (49) corresponds to the known floor for FSK systems with frequency discriminator [20, eqs.(24) and (25)]. We emphasize, however, that tight IF filtering generates much higher bit error rate floors, as demonstrated by numerical results.

In Fig. 6–10 the curves labeled with $\Delta \nu T = 0$ are the performance curves in the absence of phase noise as they can be found in the literature [1], [2] and are superimposed for comparison. With phase noise, performance is similar for low values of $E_b/N_0$, until curves reach a point where they depart from the reference curve $\Delta \nu T = 0$ to reach a bit error rate floor, as said. For $\Delta \nu T = 0$ clicks are known to give a significant contribution to the probability of bit error [2], especially for $h = 1$. When phase noise is present, the behavior of the curves in the transition region, which is from the point where they depart from the baseline $\Delta \nu T = 0$ to reach the floor level, turns out to be dictated mainly by the continuous term $P_{\text{cont}}$ in (42), rather than $P_{\text{click}}$. Of all systems, the most sensitive to phase noise is the MSK system ($h = 0.5$). Even with a narrow linewidth of $10^{-3}$ times the bit rate and $BT = 2$, for $P_b = 10^{-9}$, the penalty with respect to the case $\Delta \nu T = 0$ is about 1 dB.

The specific effect of carrier envelope fluctuations due to phase noise is relevant for narrow filters in the transition region, as can be seen in Fig. 8 to 10 by comparing the solid curves obtained from our analysis to the dotted ones obtained for the same set of parameters when this FM/AM conversion effect is not accounted for. This latter result corresponds to the case when the simplifying approach of very large (with respect to carrier spectrum) IF bandwidth is taken. It can be seen that a much smoother transition to the floor is generated when carrier
envelope fluctuations due to phase noise are properly accounted for. This corresponds to a performance degradation right in the operating SNR region, as a prescribed value of $P_b$ (such as $P_b = 10^{-9}$, for example) is achieved, with a safe margin of few decades of $P_b$ against the floor, with larger values of $E_b/N_0$, even of some dBs in certain cases.

When the bandwidth is increased, the effect of carrier envelope fluctuations gets less pronounced, as expected, as can be seen comparing Fig. 8 and 9 which refer to the case $BT = 2$ and $BT = 1$ respectively, the other parameters having the same values.

In Fig. 7, simulation results obtained from $BOSS^{TM}$ for the case $\Delta\nu T = 0.03$ have been superimposed, showing an excellent agreement with the corresponding theoretical curve and confirming the relevance of the FM/AM conversion effects in dictating the overall performance.

In Fig. 11 and 12 the required $E_b/N_0$ needed to achieve a prescribed bit error probability ($P_b = 10^{-9}$) is reported versus IF bandwidth $B$ and modulation index $h$, respectively, for various linewidths. These plots show how sensitive the system is to mismatching of IF bandwidth and modulation index. If a bandwidth smaller than the optimum value is chosen, the system suffers a penalty larger than that generated by choosing a value larger than optimum. In Fig. 7 it can be seen that the optimum bandwidth is an increasing function of the linewidth and ranges, for $h = 1$, from $BT \simeq 1.4$ when $\Delta\nu T = 0.01$, to $BT \simeq 2.2$ when $\Delta\nu T = 0.04$. Fig. 12 shows a similar sensitivity as regards modulation index. For $BT = 1.5$ the optimum values range from $h \simeq 0.75$ when $\Delta\nu T = 0.01$, to $h \simeq 1$ when $\Delta\nu T = 0.03$. By increasing the bandwidth the sensitivity curves for different linewidths tend to overlap each other onto the sensitivity curve for $\Delta\nu T = 0$ as, for large bandwidths, the main cause of errors is the shot-noise rather than the phase noise. On the contrary, an increase in the modulation index, with fixed bandwidth, enhances the envelope fluctuations due to phase noise, so, by increasing $h$ the sensitivity curves do not overlap each other.

For a given linewidth, by varying the bandwidth and the modulation index, we obtained an ensemble of curves whose envelope is the best performance curve for the system with that linewidth. These envelope curves, shown for four values of the linewidth in Fig. 13, represent the ultimate limit to performance of a heterodyne optical FSK receiver with limiter-discriminator detection and pre- and I&D post-detection filtering. In the same figure graphs are plotted, which show the values of optimum $B$
and \( h \) needed to obtain that performance. The figure is to be read in the following way. For a given bit error probability \( P_b \) and a given linewidth \( \Delta \nu T \) enter \( P_b \) from the left, go to the \( P_b \) vs. \( E_b/N_0 \) curve corresponding to \( \Delta \nu T \) and read out on the horizontal axis the lowest \( E_b/N_0 \) needed to achieve that \( P_b \). In correspondence of this \( E_b/N_0 \) go up to the \( BT \) and \( h \) curves and read out the corresponding optimum values on the right scale. For comparison, in the same figure the shot-noise-limit curve is reported, which is the best performance curve in the absence of phase noise for this system.

As expected, the limiter-discriminator-integrator detector turns out to be very sensitive to phase noise with respect, for instance, to an envelope detector. With \( \Delta \nu T = 0.16 \) the envelope detector suffers only 0.5 dB degradation [7], while our system with \( \Delta \nu T = 0.1 \) shows till 5 dB penalty with respect to the shot-noise-limit.

Fig. 13 reveals that with a right choice of bandwidth and modulation index, for \( \Delta \nu T = 0.03 \) the system suffers only about 2 dB degradation from shot noise limit. However, care must be used as being these curves ultimate limit performance curves, they do not give information about the margin from the floor. For example, if we want \( P_b = 10^{-9} \) with \( \Delta \nu T = 0.03 \), Fig. 13 says that we achieve this \( P_b \) with the minimum \( E_b/N_0 \approx 17.5 \) dB by using \( BT \approx 1.9 \) and \( h \approx 1.1 \). But this performance is achieved with \( P_b \) near the floor, so increasing \( E_b/N_0 \) will not lower significantly \( P_b \) unless we change \( B \) or \( h \).

**VIII. Conclusions**

A theoretical analysis of coherent optical FSK performances with limiter-discriminator detection and I&D post-detection filter has been performed. Effects of phase noise, intersymbol interference and intermediate-frequency filtered carrier envelope fluctuations were taken into account. It has been found that the envelope fluctuations induced by phase noise generate a performance degradation of even some dBs in certain cases and cause a smoother transition to the floor. The contribution of clicks to this degradation has been found to be negligible. However the last observation must be qualified to hold for \( h \leq 1 \), as for \( h > 1 \) it is not possible to separate clearly the click from the continuos contribution but certainly the former is more important.

Limiter-discriminator-integrator detector is sensitive to phase noise, but, with narrow linewidths of some percent of the bit rate and with a right choice of bandwidth and modulation index, it is possible to limit the performance degradation to few dBs (2 to 3). For these linewidths the optimum modulation indexes
and optimum IF bandwidths have been determined, providing
the ultimate performance of the limiter-discriminator receiver.
At \( P_b = 10^{-9} \) optimum values of \( h \) range from 0.8 for \( \Delta T = 0.01 \) to 1.75 for \( \Delta T = 0.10 \). Corresponding values of optimum IF bandwidth are 1.4 and 3.7, respectively. These values set the operating point close to the floor, and slightly larger values of \( h \) and \( B \) may be recommended in practice to allow for a safety margin.

**APPENDIX A**

**EVALUATION OF MSE BETWEEN ACTUAL AND LINEARLY FILTERED OUTPUT FREQUENCY**

In this Appendix we evaluate the mean square error between
the actual output frequency of the IF filter and its linearly filtered
version when an "alternating one-zero" bit pattern is present
with white Gaussian frequency noise at the input.

As shown in [11] the output phase \( \alpha(t) \) in (7) can be written
as in (12). We will consider only the dominant distortion term,
I.e. \(-1/2 f_3(t)\). From [11]

\[
 f_3(t) = \int_{-\infty}^{\infty} h(\tau) [\phi_S(t - \tau) - \alpha_S(t)]^3 d\tau \\
 + \phi_N(t - \tau) - \alpha_N(t)]^3 d\tau \\
 = \int_{-\infty}^{\infty} h(\tau) \left\{ [\phi_S(t - \tau) - \alpha_S(t)]^3 \\
 + 3 [\phi_S(t - \tau) - \alpha_S(t)]^2 [\phi_N(t - \tau) - \alpha_N(t)] \\
 + 3 [\phi_S(t - \tau) - \alpha_S(t)] [\phi_N(t - \tau) - \alpha_N(t)]^2 \\
 + [\phi_N(t - \tau) - \alpha_N(t)]^3 \right\} d\tau
\]  

(A.1)

where \( \phi_S(t) \) is a triangular pulse train whose Fourier series
expansion can be written as

\[
 \phi_S(t) = \sum_{n=-\infty}^{\infty} S_n e^{j2\pi n f_1 t}
\]  

(A.2)

with \( f_1 = 1/2T \) and

\[
 S_n = \begin{cases} 
 0 & n \text{ even;} \\
 \frac{2h}{\pi n^2} & n \text{ odd;}
\end{cases}
\]  

(A.3)

\( \alpha_S(t) \) is its filtered version

\[
 \alpha_S(t) = \int_{-\infty}^{\infty} h(\tau) \phi_S(t - \tau) d\tau
\]

\[
 = \sum_{n=-\infty}^{\infty} S_n H(n f_1) e^{j2\pi n f_1 t}
\]  

(A.4)

\( \phi_N(t) \) is the input phase noise, and \( \alpha_N(t) \) its corresponding
filtered version

\[
 \alpha_N(t) = \int_{-\infty}^{\infty} h(\tau) \phi_N(t - \tau) d\tau.
\]  

(A.5)

Denoting by a dot the derivative of a function, the normalized
frequency MSE between \( \dot{\alpha}(t) \) and its linearly filtered
version \( \dot{\alpha}_L(t) \) \( \dot{\alpha}_L(t) \) \( \dot{\alpha}(t) \) and \( \alpha_N(t) \) can be approximated as follows

\[
 E\left\{ \frac{[\dot{\alpha}(t) - \dot{\alpha}_L(t)]^2}{E \{\dot{\alpha}_L^2(t)\}} \right\} \approx \frac{1}{P_L} \int_{-\infty}^{\infty} W(f) df
\]  

(A.6)

where \( W(f) \) is the spectral density of \(-\dot{f}_3(t)\), and \( P_L \) is the
power of \( \dot{\alpha}_L(t) \), which is easily found to be

\[
 P_L = \int_{-\infty}^{\infty} (2\pi f)^2 W_{\phi_N}(f) |H(f)|^2 df
\]  

\[ 
 + \sum_{n=-\infty}^{\infty} |2\pi n f_1 S_n H(n f_1)|^2 
\]  

(A.7)

Here \( W_{\phi_N}(f) \) the spectral density of \( \phi_N(t) \), which is modeled
as that corresponding to a Gaussian frequency noise with constant
power spectral density \( \Delta T/2\pi \) and bandwidth \( B \), so that
\( W_{\phi_N}(f) = \Delta T/(2\pi f^2) \) for \( |f| \leq B \). When evaluating
the MSE we will let \( B \to \infty \). The evaluation of \( W(f) \) is
quite involved and we report here only the results. Following
the procedure of [11] we found that

\[
\]  

(A.8)

where the summation terms are spectra or cross spectra of the
derivatives of the RHS terms of (A.1), identified by ordering
subscripts. By letting \( H_n \equiv H(n f_1) \) to simplify notation, we have

\[
 W_1(f) = \frac{1}{36} \sum_{n=-\infty}^{\infty} (2\pi n f_1)^2 \delta(f - n f_1)
\]  

\[ 
 \cdot \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} S_k S_q S_{n-k-q} \left( H_n + 2H_k H_q H_{n-k-q}
\right.
\]  

\[ 
\left. + H_{k+q} H_{n-k-q} - H_{k+q} H_{n-k} - H_k H_{n-q} \right|^2
\]  

(A.9)

\[
 W_2(f) = \frac{1}{4} \sum_{k=-\infty}^{\infty} (2\pi f)^2 W_{\phi_N}(f - k f_1)
\]  

\[ 
 \cdot \sum_{q=-\infty}^{\infty} S_k S_{k-q} \left( H_k^* H(f - k f_1) - H(f - k f_1)
\right.
\]  

\[ 
+ H_k^* H(f - k f_1 - q f_1) + H_k^* H(f - k f_1 + q f_1)
\]  

\[ 
\left. - 2H_k^* H_{k-q} H(f - k f_1) \right|^2
\]  

(A.10)

\[
 W_3(f) = \frac{1}{4} \sum_{k=-\infty}^{\infty} (2\pi f)^2 S_k^2 \int_{-\infty}^{\infty} d\rho W_{\phi_N}(\rho) W_{\phi_N}(f - \rho - k f_1)
\]  

\[ 
\cdot \left| 2H_k^* H(\rho) H(f - \rho - k f_1) + H(f - k f_1)
\right.
\]  

\[ 
- H_k^* H(f - k f_1) - H(\rho) H(f - \rho - k f_1)
\]  

\[ 
\left. - H(f - \rho - k f_1) H(\rho - k f_1) \right|^2
\]  

(A.11)
The above expressions in (A.9) through (A.14) have been used to numerically evaluate the normalized frequency MSE in (A.6), shown in Fig. 4.

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